On the Integration of the Formula $\int \frac{dx \log x}{\sqrt{1-xx}}$ extended from $x = 0$ to $x = 1$ *

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1. The most natural method to treat formulas of the kind $\int pdx \log x$ is to reduce them to forms of the kind $\int qdx$, in which the letter q is an algebraic function of x, since the rules for integration are mainly accommodated to such formulas. But a reduction of this kind has no difficultly, whenever the function p is of such a nature that the integral $\int pdx$ can be exhibited algebraically. For, if it was $\int pdx = P$ so that the formula $\int dp \log x$ is propounded, it is immediately reduced to this expression

$$P \log x - \int \frac{Pdx}{x}$$

and so the whole task has already been reduced to the integration of the formula $\int \frac{Pdx}{x}$. But whenever the formula $\int pdx$ does not admit an algebraic integration, as it happens for our propounded formula $\int \frac{dx \log x}{\sqrt{1-xx}}$, such a reduction is not successful. For, since $\int \frac{dx}{\sqrt{1-xx}} = \arcsin x$, this reduction would give

$$\int \frac{dx \log x}{\sqrt{1-xx}} = \arcsin x \log x - \int \frac{dx}{x} \arcsin x$$

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$$\int \frac{dx \log x}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x=0 \\ \text{to} & x=1 \end{array} \right] = \frac{\pi}{2} \log 2,$$

this integral expression is to be considered to be even more remarkable, since its investigation is not obvious at all; hence I figured it to be worth one's while to have shown its truth from other sources, before I explain the method, which led me to it.

First Proof of the propounded Integration

2. Since here one mainly has to resort to infinite series, but $\log x$ does not have such a simple expansion, let us use the substitution $\sqrt{1-xx}=y$, whence $x=\sqrt{1-yy}$ and hence further

$$\log x = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.};$$

therefore, this way the propounded integral formula $\int \frac{dx \log x}{\sqrt{1-xx}}$ is transformed into the following form

$$\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.} \right),$$

where, since $y=\sqrt{1-xx}$, one has to note that the integration must be extended from $y=1$ to $y=0$; hence, if we want to permute these limits of integration, one has to change the overall sign of the form.

3. But to be less confused by such a mutation of the sign, let us denote the value in question by S that

$$S = \int \frac{dx \log x}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x=0 \\ \text{to} & x=1 \end{array} \right],$$

and after the substitution $y=\sqrt{1-xx}$, as we just mentioned, we will have

$$S = -\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.} \right) \left[\begin{array}{ll} \text{from} & y=0 \\ \text{to} & y=1 \end{array} \right]$$

But for these limits of integration, of course from $y = 0$ to $y = 1$, it is well-known that the single parts which occur here, are reduced to the following values:

$$\int \frac{yydy}{\sqrt{1-yy}} = \frac{1}{2} \cdot \frac{\pi}{2}$$
$$\int \frac{y^4dy}{\sqrt{1-yy}} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}$$
$$\int \frac{y^6dy}{\sqrt{1-yy}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}$$
$$\int \frac{y^8dy}{\sqrt{1-yy}} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$
$$\int \frac{y^{10}dy}{\sqrt{1-yy}} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2}$$
$$\text{etc.,}$$

where obviously $\frac{\pi}{2} = \int \frac{dy}{\sqrt{1-yy}}$ so that $1 : \pi$ expresses the ratio of the diameter to the circumference of the circle.

4. Therefore, if we introduce these single values, we will obtain the following infinite series for the value in question

$$S = -\frac{\pi}{2}\left(\frac{1}{2^2} + \frac{1 \cdot 3}{2 \cdot 4^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8^2} + \text{etc.}\right)$$

and so now the whole task is reduced to this that the sum of this infinite series is investigated; this might not seem to be less work than that what we had to prosecute initially. But nevertheless, it will be easy for us to get to the cognition of this series as follows.

5. Since

$$\frac{1}{\sqrt{1-zz}} = 1 + \frac{1}{2}zz + \frac{1 \cdot 3}{2 \cdot 4}z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^6 + \text{etc.,}$$

if we multiply by $\frac{dz}{z}$ and integrate both sides, we will obtain

$$\int \frac{dz}{z\sqrt{1-zz}} = \log z + \frac{1}{2^2}zz + \frac{1\cdot 3}{2\cdot 4^2}z^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2}z^6 + \text{etc.},$$

and so we have been led to our series, whose value therefore has to be found from the expression $\int \frac{dz}{z\sqrt{1-zz}} - \log z$, of course having taken the integral in such a way, that it vanishes for $z=0$; having done this set $z=1$ and the propounded series will result, i.e.

$$\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}$$

Therefore, this way the whole task has been reduced to the integral formula $\int \frac{dz}{z\sqrt{1-zz}}$, which having put $\sqrt{1-zz}=v$ goes over into the form $\frac{-dv}{1-vv}$, whose integral is known to be

$$-\frac{1}{2}\log\frac{1+v}{1-v} = -\log\frac{1+v}{\sqrt{1-vv}}.$$

If we substitute the value $\sqrt{1-zz}$ for v again, the whole expression we are after reads as follows:

$$\int \frac{dz}{z\sqrt{1-zz}} - \log z = -\log\frac{1+\sqrt{1-zz}}{z} - \log z + C = C - \log\left(1+\sqrt{1-zz}\right),$$

where the constant must be taken in such a way that the value vanishes for $z=0$, and hence it will be $C=\log 2$. Therefore, having put $z=1$ the sum in question will be $\log 2$ and hence the value of the propounded integral formula will be

$$\int \frac{dx\log x}{\sqrt{1-xx}} = S = -\frac{\pi}{2}\log 2,$$

precisely as I had found by a completely different method, from which it is already clearly seen that this truth is of greater profundity and hence worth of the mathematicians' attention.

Another Proof of the propounded Integration

6. Since $\frac{dx}{\sqrt{1-xx}}$ is the element of the circle, whose sine is $=x$, let us put this angle $=\varphi$, so that $x=\sin\varphi$ and $\frac{dx}{\sqrt{1-xx}}=d\varphi$, and after this substitution the value of the quantity S we want to find will be represented this way

$$S = \int d\varphi \log \sin \varphi \left[\begin{array}{ll} \text{from} & \varphi = 0 \\ \text{to} & \varphi = 90^\circ. \end{array} \right]$$

For, since the limits of integration had been $x = 0$ and $x = 1$, they now become $\varphi = 0$ and $\varphi = 90^\circ$ or $\varphi = \frac{\pi}{2}$. Therefore, here the whole task reduces to this that the formula $\log \sin \varphi$ is conveniently converted into an infinite series. For this purpose, let us put $\log \sin \varphi = s$ and it will be $ds = \frac{d\varphi \cos \varphi}{\sin \varphi}$. But we know that

$$\frac{\cos \varphi}{\sin \varphi} = 2 \sin 2\varphi + 2 \sin 4\varphi + 2 \sin 6\varphi + 2 \sin 8\varphi + \text{etc.}$$

For, if we multiply by $\sin \varphi$ on both sides, because of

$$2 \sin n\varphi \sin \varphi = \cos(n-1)\varphi - \cos(n+1)\varphi$$

it obviously results

$$\begin{aligned} \cos \varphi = \cos \varphi + \cos 3\varphi + \cos 5\varphi + \cos 7\varphi + \cos 9\varphi + \text{etc.} \\ - \cos 3\varphi - \cos 5\varphi - \cos 7\varphi - \cos 9\varphi - \text{etc.} \end{aligned}$$

Therefore, using this series for $\frac{\cos \varphi}{\sin \varphi}$, it will be

$$s = C - \cos 2\varphi - \frac{1}{2} \cos 4\varphi - \frac{1}{3} \cos 6\varphi - \frac{1}{4} \cos 8\varphi - \frac{1}{5} \cos 10\varphi - \text{etc.},$$

where, since $s = \log \varphi$ and hence $s = 0$, whenever $\sin \varphi = 1$ and hence $\varphi = \frac{\pi}{2}$, the constant C must be determined in such a way that for $\varphi = \frac{\pi}{2} = 90^\circ$ we have $s = 0$, from which one concludes that

$$C = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -\log 2.$$

7. Therefore, since

$$\log \sin \varphi = -\log 2 - \cos 2\varphi - \frac{1}{2} \cos 4\varphi - \frac{1}{3} \cos 6\varphi - \frac{1}{4} \cos 8\varphi - \text{etc.},$$

the value of the propounded formula will be

$$\int d\varphi \log \sin\varphi = C - \varphi \log 2 - \frac{1}{2}\sin 2\varphi - \frac{1}{8}\sin 4\varphi - \frac{1}{18}\sin\varphi$$
$$-\frac{1}{32}\sin 8\varphi - \frac{1}{50}\sin 10\varphi - \text{etc.};$$

since this expression must vanish for $\varphi = 0$, this constant entering here will be $C = 0$, so that in general

$$\int d\varphi \log \sin\varphi = -\varphi \log 2 - \frac{2\sin 2\varphi}{2^2} - \frac{2\sin 4\varphi}{4^2} - \frac{2\sin 6\varphi}{6^2} - \frac{2\sin 8\varphi}{8^2}$$
$$-\frac{2\sin 10\varphi}{10^2} - \frac{2\sin 12\varphi}{12^2} - \text{etc.}$$

If we now take $\varphi = 90^\circ = \frac{\pi}{2}$, the sine of all the angles 2φ, 4φ, 6φ, 8φ, which occur here, vanish and the value in question will be

$$S = \int d\varphi \log \sin\varphi \left[\begin{array}{ll} \text{from} & \varphi = 0 \\ \text{to} & \varphi = 90^\circ \end{array} \right] = -\frac{\pi}{2}\log 2,$$

as we demonstrated in the first proof.

8. But this proof is much better than the preceding one, since it not only provides us with the value of the propounded formula in the case 90°, but also shows its true value, whatever value is taken for φ, what can be transferred to the initially propounded formula $\int \frac{dx \log x}{\sqrt{1-xx}}$, whose value we will therefore be able to assign for each arbitrary value of x. For, if we would desire the value of this formula from $x = 0$ to $x = a$, find the angle α, whose sine is equal to a, and one will always have

$$\int \frac{dx \log x}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & \varphi = 0 \\ \text{to} & \varphi = a \end{array} \right] = -\alpha \log 2 - \frac{2\sin 2\alpha}{2^2} - \frac{2\sin 4\alpha}{4^2} - \frac{2\sin 6\alpha}{6^2} - \frac{2\sin 8\alpha}{8^2} - \text{etc.}$$

Hence it is plain, if it was $\alpha = \frac{i\pi}{2}$, while i denotes an arbitrary integer number, since all sines vanish, that the value of the formula can be expressed finitely by $-\frac{i\pi}{2}\log 2$ in these cases; in all other cases on the other hand the value of our formula will be expressed by a nice series. So, if one takes $a = \frac{1}{\sqrt{2}}$ that $\alpha = \frac{\pi}{4}$, the value of our formula will be

$$-\frac{\pi}{4}\log 2-\frac{2}{2^2}+\frac{2}{6^2}-\frac{2}{10^2}+\frac{2}{14^2}-\frac{2}{18^2}+\frac{2}{22^2}-\text{etc.},$$

which series can expressed more elegantly this way

$$-\frac{\pi}{4}\log 2-\frac{1}{2}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\text{etc.}\right),$$

and so the memorable series

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\frac{1}{121}+\text{etc.}$$

occurs, whose sum can not be reduced to known measures by any means at this point.

9. Since a so extraordinary series appeared here against all expectation, let us also expand other more beautiful cases and let us take $a=\frac{1}{2}$ so that $a=30^\circ=\frac{\pi}{6}$, and in this case the value of our formula will be

$$-\frac{\pi}{6}\log 2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.},$$

which expression can be exhibited this way

$$-\frac{\pi}{6}\log 2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right),$$

in which series the squares of three are missing.
In like manner let us now take $a=\frac{\sqrt{3}}{2}$ that $\alpha=60^\circ=\frac{\pi}{3}$, and the value of our formula in this case will result to be

$$-\frac{\pi}{3}\log 2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

or will be expressed this way

$$-\frac{\pi}{3}\log 2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right),$$

Still another Proof of the propounded Integration

10. Introduce the angle φ, whose cosine is $= x$, into our formula, or let $x = \cos\varphi$ and our formula will obtain the form $-\int d\varphi \log\cos\varphi$, which integral is to be extended from $\varphi = 90^\circ$ to $\varphi = 0$. But if we permute these limits, the value S we are looking for will be expressed this way

$$S = \int d\varphi \log\cos\varphi \begin{bmatrix} \text{from} & \varphi = 0 \\ \text{to} & \varphi = 90^\circ \end{bmatrix}.$$

To convert $\log\cos\varphi$ into a suitable series, let us as before set $s = \log\cos\varphi$ and it will be $ds = -\dfrac{d\varphi \sin\varphi}{\cos\varphi}$. But it is known that by means of a series

$$\frac{\sin\varphi}{\cos\varphi} = 2\sin 2\varphi - 2\sin 4\varphi + 2\sin 6\varphi - 2\sin 8\varphi + \text{etc.}$$

Therefore, since in general

$$2\sin n\varphi \cos\varphi = \sin(n+1)\varphi + \sin(n-1)\varphi,$$

if we multiply by $\cos\varphi$ on both sides, it will result

$$\begin{aligned} \sin\varphi = {} & \qquad\qquad \sin 3\varphi - \sin 5\varphi + \sin 7\varphi - \sin 9\varphi + \text{etc.} \\ & + \sin\varphi - \sin 3\varphi - \sin 5\varphi + \sin 7\varphi - \sin 9\varphi + \text{etc.}; \end{aligned}$$

hence, since $ds = -\frac{d\varphi \sin\varphi}{\cos\varphi}$, it will now be

$$s = C + \frac{\cos 2\varphi}{1} - \frac{\cos 4\varphi}{2} + \frac{\cos 6\varphi}{3} - \frac{\cos 8\varphi}{4} + \frac{\cos 10\varphi}{5} - \text{etc.}$$

Therefore, since $s = \log\cos\varphi$, it is evident that for $\varphi = 0$ it has to be $s = 0$, whence it follows

$$C = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -\log 2,$$

and so it will be

$$\log\cos\varphi = -\log 2 + \frac{\cos 2\varphi}{1} - \frac{\cos 4\varphi}{2} + \frac{\cos 6\varphi}{3} - \frac{\cos 8\varphi}{4} + \text{etc.},$$

which series multiplied by $d\varphi$ and integrated yields

$$S = \int d\varphi \log \cos \varphi = C - \varphi \log 2 + \frac{\sin 2\varphi}{2} - \frac{\sin 4\varphi}{8} + \frac{\sin 6\varphi}{18} - \frac{\sin 8\varphi}{32} + \frac{\sin 10\varphi}{50} - \text{etc.},$$

since which expression vanishes for $\varphi = 0$, it is plain that $C = 0$ and so we will have

$$\int d\varphi \log \cos \varphi = -\varphi \log 2 + \frac{1}{2}\left(\frac{\sin 2\varphi}{1} - \frac{\sin 4\varphi}{2^2} + \frac{\sin 6\varphi}{3} - \frac{\sin 8\varphi}{4^2} + \frac{\sin 10\varphi}{5^2} - \text{etc.}\right)$$

Therefore, having taken $\varphi = \frac{\pi}{2} = 90^\circ$, then, as before, $S = -\frac{\pi}{2}\log 2$ results. Furthermore, the integral can hence also be extended to each arbitrary limit.

11. If we subtract the last formula from the preceding one, we will in general obtain this integration

$$\int d\varphi \log \tan \varphi = -\sin 2\varphi - \frac{1}{3^2}\sin 6\varphi - \frac{1}{5^2}\sin 10\varphi - \text{etc.},$$

whence it is plain that this integral vanishes in the cases $\varphi = 90^\circ$ and is general $\varphi = \frac{i\pi}{2}$. Therefore, after we demonstrated that integration in three ways, I will explain the analysis which led me to it.

1 Analysis leading to the Integration of the Formula $\int \frac{dx \log x}{\sqrt{1-xx}}$ and other similar ones

12. This whole analysis is based on the following lemma I once proved: For the sake of brevity having put

$$(1-x^n)^{\frac{m-n}{n}} = X,$$

if hence these two integral formulas are formed

$$\int Xx^{p-1}dx \quad \text{and} \quad \int Xx^{q-1}dx,$$

which are to be extended from $x = 0$ to $x = 1$, the ratio of these values can be reduced to an infinite product as follows

$$\frac{\int Xx^{p-1}dx}{\int Xx^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n))}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

where obviously the single factors so of the numerator as of the denominator are continuously increased by the same quantity n. Here it is to be noted that this theorem can only hold, if the single letters m, n, p and q denote positive numbers, which can nevertheless always be considered as integers.

13. Concerning these two integral formulas extended from $x = 0$ to $x = 1$, two cases, in which the integration actually succeeds and the true value can be assigned absolutely, are especially worth mentioning. The first case occurs, when $p = n$, so that the formula is $\int Xx^{n-1}dx$. For, having put $x^n = y$ it will be

$$X = (1-y)^{\frac{m-n}{n}} \quad \text{and} \quad x^{n-1}dx = \frac{1}{n}dy$$

and so this formula will become $\frac{1}{n}\int dy(1-y)^{\frac{m-n}{n}}$, likewise to be extended from $y = 0$ to $y = 1$, which further, having put $1 - y = z$ goes over into the formula $-\frac{1}{n}\int z^{\frac{m-n}{n}}dz$ to be extended from $z = 1$ to $z = 0$; therefore, the value of the integral manifestly is $-\frac{1}{m}z^{\frac{m}{n}} + \frac{1}{m}$, whence the value for $z = 0$ will be $= \frac{1}{m}$. As a logical consequence for the case $p = n$ we will have

$$\int Xx^{n-1}dx \begin{bmatrix} \text{from} & x = 0 \\ \text{to} & x = 1 \end{bmatrix} = \frac{1}{m}$$

and so, if it was either $p = n$ or $q = n$, the integral becomes known absolutely.

14. In the other remarkable case we have $p = n - m$ so that the integral formula becomes $\int Xx^{n-m-1}dx$; for, then, if one sets $x(1-x)^{\frac{-1}{n}}$ or $\frac{x}{(1-x^n)^{\frac{1}{n}}} = y$, having put $x = 0$ we will find $y = 0$, but having put $x = 1$ we will find $y = \infty$; but then it will be

$$y^{n-m} = \frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}} = Xx^{n-m},$$

whence the formula to be integrated will be $\int y^{n-m}\frac{dx}{x}$. Therefore, since $\frac{x}{(1-x^n)^{\frac{1}{n}}}$, whose differential yields $\frac{dx}{x} = \frac{dy}{y(1+y^n)}$, having substituted which value our formula which is to be integrated will be

$$\int \frac{y^{n-m-1}dy}{1+y^n}$$

to be extended from $y = 0$ to $y = \infty$, which formula is remarkable, since it does not contain any irrationality anymore.

15. Therefore, since in this case we have been led to a rational formula, it is known from the elements of integral calculus that its integration can always be achieved by logarithms and circular arcs; but then for this case not so long ago I showed that the integral of this formula $\int \frac{x^{m-1}dx}{1+x^n}$ extended from $x = 0$ to $x = \infty$ is reduced to the value $\frac{\pi}{n \sin \frac{m\pi}{n}}$. Therefore, after the application, for our case we will have

$$\int \frac{y^{n-m-1}dy}{1+y^n} = \frac{\pi}{n \sin \frac{(n-m)\pi}{n}} = \frac{\pi}{n \sin \frac{m\pi}{n}};$$

therefore, for the case $p = n - m$ the value of the integral can be expressed absolutely as follows and it will be

$$\int X x^{n-m-1} dx \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = \frac{\pi}{n \sin \frac{m\pi}{n}},$$

which is also true, if $q = n - m$.

16. Having mentioned these things in advance, for the sake of brevity, let us further put

$$\int X x^{p-1} dx \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = P \quad \text{and} \quad \int X x^{q-1} \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = Q$$

and the mentioned lemma gives us this equation

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.}$$

Therefore, taking logarithms we deduced from this

$$\begin{aligned} \log P - \log Q = {} & \log(m+p) - \log p + \log(m+p+n) - \log(p+n) + \log(m+p+2n) - \log(p+2n) + \text{etc.} \\ & + \log q - \log(m+q) + \log(q+n) - \log(m+q+n) + \log(q+2n) - \log(m+q+2n) + \text{etc.} \end{aligned}$$

and this equation always holds, whatever values are attributed to the letters m, n, p and q, if they were just positive.

17. Therefore, since this equality holds in general, it will also be true, whenever some of the letters m, n, p and q are interchanged or considered as variables. Hence let us consider only the quantity p as a variable, so that the remaining letters m, n and q remain constant, and hence also Q will be a constant quantity while p varies; from this, by differentiating we will obtain this equation

$$\frac{dP}{P} = \frac{dp}{m+p} - \frac{dp}{p} + \frac{dp}{m+p+n} - \frac{dp}{p+n} + \frac{dp}{m+p+2n} - \frac{dp}{2n} + \frac{dp}{m+p+3n} - \frac{dp}{p+3n} + \text{etc.},$$

where the whole task reduces to find out, how the differential of the formula P, which is the integral, must be expressed.

18. Therefore, since P is an integral formula involving only the quantity x as variable, since in its integration the exponent p has to be considered as a constant, the quantity P can just after the integration be considered as a function of the two variables x and p, whence the question reduces to this, how the value usually expressed by the character $\left(\frac{dP}{dp}\right)$ must be investigated; if it is indicated by the letter Π, the equation found before will have this form

$$\frac{\Pi}{P} = \frac{1}{m+p} - \frac{1}{p} + \frac{1}{m+p+n} - \frac{1}{p+n} + \frac{1}{m+p+2n} - \frac{1}{p+2n} + \text{etc.}$$

But this infinite series can easily be reduced to a finite expression this way. Put

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

so that, having put $v = 1$, the letter s gives us the value $\frac{\Pi}{P}$ in question; but a differentiation on the other hand will give us

$$\frac{ds}{dv} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.},$$

the sum of which infinite series obviously is

$$\frac{v^{m+p-1}-v^{p-1}}{1-v^n}=\frac{v^{p-1}(v^m-1)}{1-v^n}.$$

Therefore, hence vice versa we conclude that it will be

$$s=\int\frac{v^{p-1}(v^m-1)dv}{1-v^n},$$

which integral formula is to be extended from $v=0$ to $v=1$; and so we will have

$$\frac{\Pi}{P}=\int\frac{v^{p-1}(v^m-1)dv}{1-v^n}\begin{bmatrix}\text{from} & v=0\\ \text{to} & v=1\end{bmatrix}.$$

19. But to investigate the value $\left(\frac{dP}{dp}\right)$, which we denoted by the letter Π here, it is well-known from the principles of integral calculus applied to functions of two variables, that the differential of the integral formula $P=\int Xx^{p-1}dx$ to result from the variability of p only is obtained, if in the formula under the integral sign Xx^{p-1} is differentiated with respect to p only and the element dp is pulled out of the integral sign; but since X on the other hand does not contain p, it has to be considered as a constant here, but the differential of the power x^{p-1} will be $x^{p-1}dp\log x$; therefore, the final result of this differentiation will be

$$dP=dp\int Xx^{p-1}dx\log x,$$

so that the factor $\log x$ additionally entered the integral, from which it is manifest that it will be

$$\Pi=\int Xx^{p-1}dx\log x\begin{bmatrix}\text{from} & x=0\\ \text{to} & x=1\end{bmatrix};$$

hence we will be able to formulate the following general theorem.

General Theorem

20. *For the sake of brevity having put* $X=(1-x^n)^{\frac{m-n}{n}}$, *if the following integral formulas are extended from* $x=0$ *to* $x=1$, *the following equality will always be true*

$$\frac{\int Xx^{p-1}dx \log x}{\int Xx^{p-1}dx} = \int \frac{x^{p-1}(x^m - 1)dx}{1 - x^n}.$$

For, there is no obstruction that we wrote x instead of v, since these values only depend on the limits of the integration.

21. Therefore, this way we have been led to the integration of the formula $\int Xx^{p-1}dx \log x$, in which the logarithm $\log x$ is contained in the integrand as a factor, and we are now able to express the values of these formulas by ordinary integral formulas, since

$$\int Xx^{p-1}dx \log x = \int Xx^{p-1}dx \cdot \int \frac{x^{p-1}(x^m - 1)dx}{1 - x^n},$$

having extended the integration from $x = 0$ to $x = 1$, of course, where, for the sake of brevity, we put $(1 - x^n)^{\frac{m-n}{n}} = X$. Therefore, for two memorable cases explained above [par. 13 - 15] we already derived two particular theorems.

Particular Theorem 1 in which $p = n$

22. Since above [par. 13] we saw that in the case $p = n$ we have $\int Xx^{n-1}dx = \frac{1}{m}$, having substituted this value, we have this elegant equation

$$\int Xx^{n-1}dx \log x = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1)dx}{1 - x^n},$$

while both integrals are extended from $x = 0$ to $x = 1$.

Particular Theorem 2 in which $p = n - m$

23. Since for this case, in which $p = n - m$, above [par. 15] we showed that

$$\int Xx^{m-n-1}dx = \frac{\pi}{n \sin \frac{m\pi}{n}},$$

we are now led to the following most remarkable integration

$$\int Xx^{n-m-1}dx \log x = \frac{\pi}{n \sin \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m - 1)dx}{1 - x^n},$$

if both these integrals are extended from $x = 0$ to $x = 1$, of course, where still $X = (1 - x^n)^{\frac{m-n}{n}}$.

24. Here it should carefully be noted that the general theorem extends very far, since it contains three indefinite exponents, namely m, n and p, which are completely arbitrary, which can therefore be defined in infinitely may ways, as long as positive values are attributed to them, so that the value of this integral formula $\int Xx^{p-1}dx\log x$, which because of the factor $\log x$ has to be considered as transcendental quantity, can always be expressed by ordinary integral formulas; since these are most general, it will be worth one's while to expand several special cases.

I. Expansion of the Case in which $m = 1$ and $n = 2$

25. Therefore, in this case it will be $X = \frac{1}{\sqrt{1-xx}}$, whence for this case the theorem reads as follows

$$\int \frac{x^{p-1}dx\log x}{\sqrt{1-xx}} = -\int \frac{x^{p-1}dx}{\sqrt{1-xx}} \cdot \int \frac{x^{p-1}dx}{1+x},$$

if these single integrals are extended from $x = 0$ to $x = 1$. Therefore, since here only the exponent p is arbitrary, hence let us go through the following examples.

26. Therefore, in this case the above equation becomes

$$\int \frac{dx\log x}{\sqrt{1-xx}} = -\int \frac{dx}{\sqrt{1-xx}} \cdot int\frac{dx}{1+x},$$

where, having extended the integrals from $x = 0$ to $x = 1$, it is known to be

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2} \quad \text{and} \quad \int \frac{dx}{1+x} = \log 2,$$

so that we now have

$$\int \frac{dx\log x}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = -\frac{\pi}{2}\log 2,$$

which is the formula we treated initially and whose validity we proved in three different ways already.

27. The same value can be found from the second particular theorem, in which it was $p = n - m$, since now because of $n = 2$ and $m = 1$ it will be $p = 1$; for, hence because of $X = \dfrac{1}{\sqrt{1-xx}}$ this theorem yields

$$\int \frac{dx \log x}{\sqrt{1-xx}} = \frac{\pi}{2\sin\frac{\pi}{2}} \int -\frac{dx}{1+x} = -\frac{\pi}{2}\log 2.$$

EXAMPLE 2 IN WHICH $p = 2$

28. Therefore, in this case the above equation has this form

$$\int \frac{xdx \log x}{\sqrt{1-xx}} = -\int \frac{xdx}{\sqrt{1-xx}} \cdot \int \frac{xdx}{1+x}.$$

Now, having extended the integrals from $x = 0$ to $x = 1$, it is known to be

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 \quad \text{and} \quad \int \frac{xdx}{1+x} = 1 - \log 2,$$

so that we have

$$\int \frac{xdx \log x}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x = 0 \\ \text{to} & x = 1 \end{bmatrix} = \log 2 - 1.$$

29. Since in this formula the integral $\int \frac{xdx}{\sqrt{1-xx}}$ can be exhibited algebraically, that integral is $= 1 - \sqrt{1-xx}$, the value in question can be found by usual reductions, since

$$\int \frac{xdx \log x}{\sqrt{1-xx}} = (1 - \sqrt{1-xx}) \log x - \int \frac{dx}{x}(1 - \sqrt{1-xx});$$

and having put $x = 1$ it will be

$$\int \frac{xdx \log x}{\sqrt{1-xx}} = -\int \frac{dx}{x}(1 - \sqrt{1-xx}),$$

to integrate which formula let

$$1 - \sqrt{1-xx} = z,$$

whence one finds $xx = 2z - zz$, therefore, $2\log x = \log z + \log(2-z)$, and so it will be

$$\frac{dx}{x} = \frac{dz(1-z)}{z(2-z)},$$

having substituted which values it will be

$$+\int \frac{dx}{x}(1-\sqrt{1-xx}) = +\int \frac{dz(1-z)}{2-z},$$

which value will therefore be $= C - z - \log(2-z)$. Therefore, since having put $x = 0$ we have $z = 0$, the constant will be $C = +\log 2$; therefore, for $x = 1$, since then $z = 1$, the value of this integral will be $\log 2 - 1$, precisely as before.

30. The theorem mentioned first above, in which it was $p = n - 2$, yields the same value; for, hence immediately $\int \frac{xdx \log x}{\sqrt{1-xx}} = \int -\frac{xdx}{1+x}$. But we have seen before that $\int \frac{xdx}{1+x} = 1 - \log 2$ so that also from this the value in question results to be $\log 2 - 1$.

Example 3 in which $p = 3$

31. Therefore, in this case the equation mentioned in the general theorem will have this form

$$\int \frac{xxdx \log x}{\sqrt{1-xx}} = -\int \frac{xxdx}{\sqrt{1-xx}} \cdot \int \frac{xxdx}{1+x}.$$

But by commonly known reductions it is found to be

$$\int \frac{xxdx}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = \frac{1}{2} \cdot \frac{\pi}{2};$$

but the spurious fraction $\frac{xx}{1+x}$ is resolved into these parts

$$x - 1 + \frac{1}{1+x},$$

whence it will be

$$\int \frac{xxdx}{1+x} = \frac{1}{2}xx - x + \log(1+x),$$

which integral already vanishes from $x = 0$; therefore, for $x = 1$ its value will be $= -\frac{1}{2} + \log 2$; therefore, the integral in question will be

$$\int \frac{xxdx\log x}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = -\frac{\pi}{4}\left(\log 2 - \frac{1}{2}\right).$$

EXAMPLE 4 IN WHICH $p = 4$

32. Therefore, in this case the above equation will have this form

$$\int \frac{x^3dx\log x}{\sqrt{1-xx}} = -\int \frac{x^3dx}{\sqrt{1-xx}} \cdot \int \frac{x^3dx}{1+x}.$$

By commonly known reductions it is known to be

$$\int \frac{x^3dx}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = \frac{2}{3};$$

but then the spurious fraction $\frac{x^3}{1+x}$ is resolved into these parts

$$xx - x + 1 - \frac{1}{x+1},$$

whence by integrating

$$\int \frac{x^3dx}{1+x} = \frac{1}{3}x^3 - \frac{1}{2}xx + x - \log(1+x),$$

from which the value of the formula will be $= \frac{5}{6} - \log 2$. Therefore, having substituted these values we obtain this integration

$$\int \frac{x^3dx\log x}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = -\frac{2}{3}\left(\frac{5}{6} - \log 2\right).$$

EXAMPLE 5 IN WHICH $p = 5$

33. Therefore, in this case the above equation will have this form

$$\int \frac{x^4dx\log x}{\sqrt{1-xx}} = -\int \frac{x^4dx}{\sqrt{1-xx}} \cdot \int \frac{x^4}{1+x}.$$

But it is known to be

$$\int \frac{x^4dx}{\sqrt{1-xx}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2};$$

but then the spurious fraction $\frac{x^4}{1+x}$ is obviously resolved into these parts

$$x^3 - xx + x - 1 + \frac{1}{x+1},$$

whence by integrating

$$\int \frac{x^4 dx}{1+x} = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}xx - x + \log(1+x),$$

from which the value of the formula will be $= -\frac{7}{12} + \log 2$. Therefore, having substituted these values this integration will result

$$\int \frac{x^4 dx \log x}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = -\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} \left(\log 2 - \frac{7}{12} \right).$$

EXAMPLE 6 IN WHICH $p = 6$

34. Therefore, in this case the above equation will become

$$\int \frac{x^5 dx \log x}{\sqrt{1-xx}} = -\int \frac{x^5 dx}{\sqrt{1-xx}} \cdot \int \frac{x^5 dx}{1+x}.$$

But it is known that by usual reduction

$$\int \frac{x^5 dx}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = \frac{2 \cdot 4}{3 \cdot 5};$$

but then the spurious fraction $\frac{x^5}{1+x}$ is resolved into these parts

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

whence by integration we obtain

$$\int \frac{x^5 dx}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - \log(1+x),$$

from which the value of this formula will be $= \frac{47}{60} - \log 2$; having substituted these values the following integration will result

$$\int \frac{x^5 dx \log x}{\sqrt{1-xx}} \left[\begin{array}{ll} \text{from} & x = 0 \\ \text{to} & x = 1 \end{array} \right] = -\frac{2 \cdot 4}{3 \cdot 5} \left(\frac{47}{60} - \log 2 \right).$$

II. Expansion of the Case in which $m = 3$ and $n = 2$

35. Therefore, here it will be $X = \sqrt{1-xx}$, whence our general theorem will give us this equation

$$\int x^{p-1}dx \log x \cdot \sqrt{1-xx} = \int x^{p-1}dx\sqrt{1-xx} \cdot \int \frac{x^{p-1}(x^3-1)dx}{1-xx};$$

here, since

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

the last integral formula will be

$$-\int x^p dx - \int \frac{x^{p-1}dx}{1+x},$$

which integrated from $x = 0$ to $x = 1$ gives

$$-\frac{1}{p+1} - \int \frac{x^{p-1}dx}{1+x},$$

whence we will have

$$\int x^{p-1}dx \log x \cdot \sqrt{1-xx} = -\int x^{p-1}dx\sqrt{1-xx} \cdot \left(\frac{1}{p+1} + \int \frac{x^{p-1}dx}{1+x}\right).$$

Therefore, hence it will be helpful to have noted the following examples.

Example 1 in which $p = 1$

36. Therefore, for this case the last factor will become $\frac{1}{2} + \log 2$ so that

$$\int dx \log x \cdot \sqrt{1-xx} = -\left(\frac{1}{2} + \log 2\right) \int dx\sqrt{1-xx}.$$

But for the formula $\int dx\sqrt{1-xx}$ set

$$\sqrt{1-xx} = 1 - vx$$

and it will be $x = \frac{2v}{1+vv}$ and $\sqrt{1-xx} = \frac{1-vv}{1+vv}$ and $dx = \frac{2dv(1-vv)}{(1+vv)^2}$, whence it will be

$$dx\sqrt{1-xx} = \frac{2dv(1-vv)^2}{(1+vv)^3},$$

whose integral is resolved into these parts

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \arctan v;$$

since this expression must be extended from $x = 0$ to $x = 1$, the first limit will be $v = 0$, the other limit on the other hand is $v = 1$, so that that integral must be extended from $v = 0$ to $v = 1$. But the expression immediately vanishes for $v = 0$, but for $v = 1$ the value of the integral will be $= \frac{\pi}{4}$; therefore, we will have

$$\int dx \log x \cdot \sqrt{1-xx} \left[\begin{array}{ll} \text{from} & x=0 \\ \text{to} & x=1 \end{array} \right] = -\frac{\pi}{4}\left(\frac{1}{2} + \log 2\right).$$

37. Here we certainly presented a non straight-forward calculation; this is due the reduction to rationality of the formula $\sqrt{1-xx}$; but the formula $\int dx\sqrt{1-xx}$ itself is immediately clear to express the area of the quadrant of the circle, whose radius is $= 1$, which we know to be $= \frac{\pi}{4}$. Furthermore, one could have used this reduction

$$\int dx\sqrt{1-xx} = \frac{1}{2}x\sqrt{1-xx} + \frac{1}{2}\int \frac{dx}{\sqrt{1-xx}},$$

whose value extended from $x = 0$ to $x = 1$ manifestly gives $\frac{\pi}{4}$.

Example 2 in which $p = 2$

38. Therefore, in this case the last factor is

$$\frac{1}{3} + \int \frac{xdx}{1+x} = \frac{4}{3} - \log 2$$

and so we will have

$$\int xdx \log x \cdot \sqrt{1-xx} = -\left(\frac{4}{3} - \log 2\right)\int xdx\sqrt{1-xx};$$

but it is perspicuous to be

$$\int xdx\sqrt{1-xx} = C - \frac{1}{3}(1-xx)^{\frac{3}{2}},$$

which value extended from $x=0$ to $x=1$ yields $\frac{1}{3}$, so that we have

$$\int dx \log x \cdot \sqrt{1-xx} \left[\begin{array}{ll} \text{from} & x=0 \\ \text{to} & x=1 \end{array} \right] = -\frac{1}{3}\left(\frac{4}{3} - \log 2\right).$$

III. Expansion of the Case in which $m = 1$ and $n = 3$

39. Therefore, in this case it will be $X = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, whence the general theorem gives us this equation

$$\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}(x-1)dx}{1-x^3},$$

where the last formula is reduced to $-\int \frac{x^{p-1}dx}{xx+x+1}$, so that we have

$$\int \frac{x^{p-1}dx \log x}{\sqrt[3]{(1-x^3)^2}} = -\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}dx}{xx+x+1}.$$

Therefore, let us add the following example.

Example 1 in which $p = 1$

40. Therefore, in this case the last factor becomes $\int \frac{dx}{xx+x+1}$, whose indefinite integral is found to be $\frac{2}{\sqrt{3}} \arctan \frac{x\sqrt{3}}{2+x}$, which value, having put $x=1$, goes over into $\frac{\pi}{3\sqrt{3}}$; therefore, in this case we will have

$$\int \frac{dx \log x}{\sqrt[3]{(1-x^3)^2}} = -\frac{\pi}{3\sqrt{3}} \int \frac{dx}{\sqrt[3]{(1-x^3)^2}};$$

but the integral formula $\int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$ involves a peculiar transcendental quantity, which can expressed neither by logarithms nor circular arcs.

Example 2 in which $p = 2$

41. Therefore, in this case the second factor will be $\int \frac{xdx}{1+x+xx}$, which is resolved into these parts

$$\frac{1}{2}\int\frac{2xdx+dx}{1+x+xx}-\frac{1}{2}\int\frac{dx}{1+x+xx},$$

where the integral of the first part is

$$\frac{1}{2}\log(1+x+xx)=\frac{1}{2}\log 3 \quad \text{(having put } x=1\text{, of course),}$$

but the integral of the other part is $-\frac{1}{2}\cdot\frac{\pi}{3\sqrt{3}}$; having substituted this value, we will have

$$\int\frac{xdx\log x}{\sqrt[3]{(1-x^3)^2}}=-\frac{1}{2}\left(\log 3-\frac{\pi}{3\sqrt{3}}\right)\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

But now one can assign this integral formula conveniently by the reduction mentioned above initially; for, since here $m=1$ and $n=3$, but then we took $p=2$, it will be $p=n-m$. But above (par. 15) we found that the integral will be $=\frac{\pi}{n\sin\frac{m\pi}{n}}$ in this case, which value in our case goes over into $\frac{\pi}{3\sin\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}$. Therefore, having substituted this value we will be able to express our formula by mere known quantities this way

$$\int\frac{xdx\log x}{\sqrt[3]{(1-x^3)^2}}\begin{bmatrix}\text{from} & x=0\\ \text{to} & x=1\end{bmatrix}=-\frac{\pi}{3\sqrt{3}}\left(\log 3-\frac{\pi}{3\sqrt{3}}\right).$$

IV. Expansion of the Case in which $m=2$ and $n=3$

42. Therefore, in this case it will be $X=\frac{1}{\sqrt[3]{(1-x^3)}}$, whence the general theorem yields this equation

$$\int\frac{x^{p-1}dx\log x}{\sqrt[3]{(1-x^3)}}=\int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{p-1}(xx-1)dx}{1-x^3},$$

where the second formula is transformed into $-\int\frac{x^{p-1}dx(1+x)}{1+x+xx}$; hence it will be

$$\int\frac{x^{p-1}dx\log x}{\sqrt[3]{(1-x^3)}}=-\int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{p-1}dx(1+x)}{1+x+xx},$$

whence we want to go through the following examples.

43. Therefore, in this case the second term will be $\int \frac{dx(1+x)}{1+x+xx}$, whose integral is split into these parts

$$\frac{1}{2}\int \frac{2xdx + dx}{1+x+xx} + \frac{1}{2}\int \frac{dx}{1+x+xx},$$

whence for the case $x = 1$ manifestly $\frac{1}{2}\left(\log 3 + \frac{\pi}{3\sqrt{3}}\right)$ results; therefore, our equation will be

$$\int \frac{dx \log x}{\sqrt[3]{(1-x^3)}} = -\frac{1}{2}\left(\log 3 + \frac{\pi}{3\sqrt{3}}\right)\int \frac{dx}{\sqrt[3]{(1-x^3)}}.$$

But in this integral formula because of $m = 2$ and $n = 3$, since we took $p = 1$, it will be $p = n - m$; therefore, for this case by par. 15 the value of this integral formula can be expressed absolutely and it will be $\int \frac{dx}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}}$; as a logical consequence even in this case by absolute quantities we obtain this formula

$$\int \frac{dx \log x}{\sqrt[3]{(1-x^3)}} \begin{bmatrix} \text{from} & x = 0 \\ \text{to} & x = 1 \end{bmatrix} = -\frac{\pi}{3\sqrt{3}}\left(\log 3 + \frac{\pi}{3\sqrt{3}}\right).$$

44. If we combine this form with the last of the preceding case, which likewise resulted expressed absolutely, first their sum will give

$$\int \frac{xdx \log x}{\sqrt[3]{(1-x^3)^2}} + \int \frac{dx \log x}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi \log 3}{3\sqrt{3}},$$

but if the second is subtracted from the first, this equation will result

$$\int \frac{xdx \log x}{\sqrt[3]{(1-x^3)^2}} - \int \frac{dx \log x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi\pi}{27}.$$

Since this way we have been led to rather simple expressions, it will be worth one's while to represent both equations in another form, in which both parts of the integral can conveniently be combined into one; let us set

$$\frac{x}{\sqrt[3]{(1-x^3)}} = z,$$

whence $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, and so the first formula will have the form $\int \frac{zzdx \log x}{x}$, but the second this form $\int \frac{zdx \log x}{x}$; but then we will have $\frac{x^3}{1-x^3} = z^3$, whence $x^3 = \frac{z^3}{1+z^3}$ and hence

$$\log x = \log z - \frac{1}{3}\log(1+z^3) = \log \frac{z}{\sqrt[3]{(1+z^3)}}$$

and hence further

$$\frac{dx}{x} = \frac{dz}{z} - \frac{zzdz}{1+z^3} = \frac{dz}{z(1+z^3)},$$

whence using these values the first integral formula becomes $\int \frac{zdz}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}}$, but the other formula will be $\int \frac{dz}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}}$.

45. But since the integrals must be extended from $x = 0$ and $x = 1$, it is to be noted that in the case $x = 0$ also $z = 0$, but for $x = 1$ we have $z = \infty$, so that these new formulas must be extended from $z = 0$ to $z = \infty$. Having observed this the first of these formulas will give

$$\int \frac{zdz}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}} \left[\begin{array}{ll} \text{from} & z=0 \\ \text{to} & z=\infty \end{array} \right] = -\frac{\pi \log 3}{3\sqrt{3}} + \frac{\pi\pi}{27},$$

the second on the other hand

$$\int \frac{dz}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}} \left[\begin{array}{ll} \text{from} & z=0 \\ \text{to} & z=\infty \end{array} \right] = -\frac{\pi \log 3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Therefore, hence the sum of these formulas will be

$$\int \frac{dz(1+z)}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi \log 3}{3\sqrt{3}},$$

but the difference on the other hand

$$\int \frac{dz(z-1)}{1+z^3} \log \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

46. It will be useful here to have noted that this logarithm $\log \frac{z}{\sqrt[3]{(1+z^3)}}$ can conveniently be converted into a rather simple infinite series; for, since

$$\log \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{1}{3}\log \frac{z^3}{1+z^3} = -\frac{1}{3}\log \frac{1+z^3}{z^3},$$

by a series it will be

$$\log \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{1}{3}\left(\frac{1}{z^3} - \frac{1}{2z^6} + \frac{1}{3z^9} - \frac{1}{4z^{12}} + \frac{1}{5z^{15}} - \text{etc.}\right);$$

but this resolution has no use for the expansion of the integrals into series, since the powers of z occur in the denominators and hence the single parts can not be integrated in such a way that they vanish for $z = 0$.

Example 2 in which $p = 2$

47. Therefore, in this case the second factor becomes $\int \frac{xdx(1+x)}{1+x+xx}$, which is split into these parts

$$\int dx - \int \frac{dx}{1+x+xx},$$

whose integral extended from $x = 0$ to $x = 1$ therefore is $= 1 - \frac{\pi}{3\sqrt{3}}$. Therefore, we are hence led to the equation

$$\int \frac{xdx\log x}{\sqrt[3]{(1-x^3)}} = -\left(1 - \frac{\pi}{3\sqrt{3}}\right)\int \frac{xdx}{\sqrt[3]{(1-x^3)}}.$$

But here it is to be noted that that integral formula can not be exhibited absolutely by any means, but involves a certain peculiar transcendental quantity.

V. Expansion of the Case in which $m = 2$ and $n = 4$

48. Therefore, in this case it will be $X = \frac{1}{\sqrt{1-x^4}}$, whence our general theorem will give us this equation

$$\int \frac{x^{p-1}dx\log x}{\sqrt{1-x^4}} = -\int \frac{x^{p-1}dx}{\sqrt{1-x^4}} \cdot \int \frac{x^{p-1}dx}{1+xx};$$

but the first particular problem on the other hand for this case yields

$$\int \frac{x^3dx \log x}{\sqrt{1-x^4}} = -\frac{1}{2}\int \frac{x^3dx}{1+xx}.$$

But since

$$\int \frac{x^3dx}{1+xx} = \frac{1}{2} - \frac{1}{2}\log 2,$$

in absolute quantities it will be

$$\int \frac{x^3dx \log x}{\sqrt{1-x^4}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = -\frac{1}{4}(1-\log 2),$$

but this case is the same as the one treated above (par. 28). For, if we put $xx = y$ here, having done which the limits of integration remain $y = 0$ and $y = 1$, it will be $\log x = \frac{1}{2}\log y$ and $xdx = \frac{1}{2}dy$; having substituted these values our equation will go over into the form $\frac{1}{4}\int \frac{ydy \log y}{\sqrt{1-yy}} = -\frac{1}{4}(1-\log 2)$ or $\int \frac{ydy \log y}{\sqrt{1-yy}} = \log 2 - 1$, precisely as above.

49. But the other particular theorem accommodated to the present case will give

$$\int \frac{xdx \log x}{\sqrt{1-x^4}} = -\frac{\pi}{4}\int \frac{xdx}{1+xx};$$

but

$$\int \frac{xdx}{1+xx} = \log\sqrt{1+xx} = \frac{1}{2}\log 2.$$

so that we have

$$\int \frac{xdx \log x}{\sqrt{1-x^4}} \begin{bmatrix} \text{from} & x=0 \\ \text{to} & x=1 \end{bmatrix} = -\frac{\pi}{8}\log 2.$$

But if here we set $xx = y$ as before, we will obtain $\int \frac{dy \log y}{\sqrt{1-yy}}$, which is the case treated above (par. 26). In these two cases the exponent p was an even number, whence it will be convenient to expand cases of odd numbers.

EXAMPLE 1 IN WHICH $p = 1$

50. Therefore, in this case the second integral formula will become

$$\int \frac{dx}{1+xx} = \arctan x,$$

so that having put $x = 1$ $\frac{\pi}{4}$ results; but then our equation will be

$$\int \frac{dx \log x}{\sqrt{1-x^4}} = -\frac{\pi}{4} \int \frac{dx}{\sqrt{1-x^4}},$$

having extended the integrals from $x = 0$ to $x = 1$, of course; here the formula $\int \frac{dx}{\sqrt{1-x^4}}$ expresses the arc of the rectangular elastic curve and can hence not be exhibited absolutely.

EXAMPLE 2 IN WHICH $p = 3$

51. Therefore, in this case the second integral formula will be

$$\int \frac{xxdx}{1+xx} = \int dx - \int \frac{dx}{1+xx},$$

whose integral having put $x = 1$ becomes $= 1 - \frac{\pi}{4}$, so that our equation now becomes

$$\int \frac{xxdx \log x}{\sqrt{1-x^4}} = -\left(1 - \frac{\pi}{4}\right) \int \frac{xxdx}{\sqrt{1-x^4}},$$

which integral likewise can not be exhibited absolutely; for, it expresses the ordinate of the rectangular elastic curve.

52. But although these two examples led to intractable formulas, nevertheless I showed recently that the product of these two integrals, $\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{xxdx}{\sqrt{1-x^4}}$ is equal to the area of the circle, whose diameter is $= 1$, or is $= \frac{\pi}{4}$; therefore, combining these two examples we obtain the remarkable theorem that

$$\int \frac{dx \log x}{\sqrt{1-x^4}} \cdot \int \frac{xxdx \log x}{\sqrt{1-x^4}} = \frac{\pi^2}{16}\left(1 - \frac{\pi}{4}\right).$$

But it is obvious that innumerable other theorems of this kind can be found from this source, which considered separately are to be considered to be of highest profundity.